Brown University

UNPUBLISHED PRELIMINARY DATA

SUFFICIENT CONDITIONS FOR STABILITY AND INSTABILITY

OF AUTONOMOUS FUNCTIONAL - DIFFERENTIAL EQUATIONS

Jack K. Hale †



† This research was supported in part by National Aeronautics and Space Administration under Contract No. NGR 40-002-015, in part by the United States Army Research Office, Durham, under Contract No. DA-31-124 ARO-D-270 and in part by the United States Air Force through the Air Force Office of Scientific Research under Contract No. AF AFOSR 693-64.

# SUFFICIENT CONDITIONS FOR STABILITY AND INSTABILITY OF AUTONOMOUS FUNCTIONAL - DIFFERENTIAL EQUATIONS

Jack K. Hale

1. Introduction and notation. An autonomous functional-differential equation with finite time lag is a generalization of the usual concept of differential equations with retarded arguments of the form

$$\dot{x}(t) = F(x(t), x(t-\tau_1), \ldots, x(t-\tau_k)),$$

where the $\tau_j$ are positive constants. In this paper, we will be concerned with the application of the concept of Lyapunov functionals to the determination of sufficient conditions for the stability of such systems. Lyapunov functionals have been applied to these equations by many authors and the reader should consult the book of Krasovskii [1] for a detailed bibliography. Lyapunov functionals can also be employed to discuss the stability of nonautonomous systems; that is, systems in which $F$ contains $t$ explicitly, but we restrict ourselves to the autonomous case since more general results are obtainable. In particular, we wish to discuss the implications of a theorem of LaSalle [2] for ordinary differential equations properly extended to functional-differential equations. Some of the results of this paper have been announced in [3], and the present paper contains the complete proofs of those results as well as some new results on instability and many applications. The extension of the results to infinite time lag is contained in section 4.

To discuss functional-differential equations in the proper setting, it is necessary to introduce some notation. $E^n$ will denote the real Euclidean space of $n$-vectors and $|x|$ will denote the norm of the vector $x$ in $E^n$. If

$r \geq 0$ is given, $C = C([-r, 0], E^n)$ will denote the space of continuous functions with domain $[-r, 0]$ and range in $E^n$. The norm in this space will be the uniform one: $\|\varphi\| = \max_{-r \leq \theta \leq 0} |\varphi(\theta)|$ for $\varphi$ in $C$. Suppose $x$ is any given function with domain $[-r, \infty)$ and range in $E^n$. For any $t \geq 0$, we will let $x_t$ denote a translation of the ~~the~~ restriction of $x$ to the interval $[t - r, t]$; more specifically, ~~that is,~~ $x_t$ is an element of $C$ defined by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$. In other words, the graph of $x_t$ is ~~the same as~~ the graph of $x$ on $[t - r, t]$ ~~except it is~~ shifted to the interval $[-r, 0]$. The reader is urged at this point to use extreme caution in distinguishing between the following symbols: $x$ is a function taking $[-r, \infty]$ into $E^n$, $x_t$ for $t \geq 0$ is a function taking $[-r, 0]$ into $E^n$, $x(t)$ is the value of $x$ at $t$ and $x_t(\theta)$ is the value of $x_t$ at $\theta$; $x(\varphi)$ also has a special meaning to be explained in a moment.

If $H$ is a given positive constant, we use the notation $C_H$ for the set $\{\varphi \text{ in } C: \|\varphi\| < H\}$; that is, $C_H$ is the open ball in $C$ of radius $H$.

If $f(\varphi)$ is a function defined for every $\varphi$ in $C_H$ and $\dot{x}(t)$ is the right hand derivative of $x(t)$, we consider the following autonomous functional-differential equation:

$$\dot{x}(t) = f(x_t), \qquad t \geq 0. \tag{1}$$

We say $x(\varphi)$ is a solution of (1) with initial condition $\varphi$ in $C_H$ at $t = 0$ if there is an $A > 0$ such that $x(\varphi)$ is a function from $[-r, A)$ into $E^n$ such that $x_t(\varphi)$ is in $C_H$ for $0 \leq t < A$, $x_0(\varphi) = \varphi$ and $x(\varphi)(t)$ satisfies (1) for $0 \leq t < A$.

In a manner quite ~~very~~ analogous to that used for ordinary differential equations, one can prove the following results: If $f(\varphi)$ is continuous in $C_H$, then for any $\varphi$ in $C_H$, there is a solution of (1) with initial condition $\varphi$ at $t = 0$. If $f(\varphi)$ is locally Lipschitzian in $\varphi$ then there is only one solution with initial condition $\varphi$ at $t = 0$ and the solution $x(\varphi)$ depends continuously upon $\varphi$. Also, $f(\varphi)$ locally Lipschitzian in $\varphi$ (or an even the weaker hypotheses that $f$ maps bounded sets into bounded sets) implies the solutions can be extended in $C$ until the boundary of $C_H$ is reached.

It is clear that the differential-difference equation discussed before is a special case of (1). However, system (1) is much more general and, in particular, could be of the form

$$f(x_t) = F(\int_{-r}^{0} x(t+\theta)d\theta).$$

If $f(0) = 0$, then the solution $x = 0$ of (1) is said to be <u>stable</u> if for every $\epsilon > 0$, there is a $\delta > 0$ such that $\|\varphi\| < \delta$ implies $x_t(\varphi)$ exists for $t \geq 0$, is in $C_H$ and $\|x_t(\varphi)\| < \epsilon$ for all $t \geq 0$. If, in addition, there is an $H_1 < H$ such that $\|\varphi\| < H_1$ implies $x_t(\varphi)$ is in $C_H$ for $t \geq 0$ and $x_t(\varphi) \to 0$ as $t \to \infty$, then the solution $x = 0$ is said to be <u>asymptotically stable</u>.

If $V$ is a continuous scalar function on $C_H$, we define $\dot{V}_{(1)}(\varphi)$ by the following relation

$$\dot{V}_{(1)}(\varphi) = \overline{\lim_{h\to 0^+}} \frac{1}{h}(V(x_h(\varphi)) - V(\varphi))$$

2. <u>Sufficient conditions for stability and instability</u>. In this section, we consider some stability and instability theorems for autonomous systems (1) along the lines of some results of LaSalle [2] on ordinary differential equations. Part of this section appeared in a paper by Hale [3].

We will always suppose that the function $f(\varphi)$ in (1) is continuous and locally Lipschitzian on $C_H$. When $H = \infty$, then $C_H = C$ and we will be speaking of global stability ($H_1 = \infty$).

Since (1) is autonomous, it is quite natural to consider system (1) as defining motions or paths in $C$. In fact, we can define a <u>motion through</u> $\varphi$ as the set of functions in $C$ given by $\bigcup_{0 \leq t < A} x_t(\varphi)$ where the interval $[-r, A)$ is the interval of definition of $x(\varphi)$. An element $\psi$ of $C$ is in $\Omega(\varphi)$, the <u>$\omega$-limit set of $\varphi$</u>, if $x(\varphi)$ is defined on $[-r, \infty)$ and there is a sequence of nonnegative real numbers $t_n$, $t_n \to \infty$ as $n \to \infty$ such that $\|x_{t_n}(\varphi) - \psi\| \to 0$ as $n \to \infty$. A set $M$ in $C$ is called an <u>invariant set</u> if for any $\varphi$ in $M$, there exists a function $x$, depending on $\varphi$, defined on $(-\infty, \infty)$, $x_t$ in $M$ for $t$ in $(-\infty, \infty)$, $x_0 = \varphi$, such that if $x^*(\sigma, x_\sigma)$ is the solution (1) with initial value $x_\sigma$ at $\sigma$, then $x^*(\sigma, x_\sigma) = x_t$ for all $t \geq \sigma$. Notice that to any element of an invariant set there corresponds a solution which must be the simplest form of system (1).

For any $H_1 < H$, there is a constant $L$ such that $|f(\varphi)| \leq L$ for all $\varphi$ with $\|\varphi\| \leq H_1$. From this, one easily obtains the following result:

<u>Lemma 1.</u> If $x(\varphi)$ is a solution of system (1) with initial function $\varphi$ at $0$, defined on $[-r, \infty)$ and $\|x_t(\varphi)\| \leq H_1 < H$ for all $t$ in $[0, \infty)$, then the family of functions $\{x_t(\varphi), t \geq 0\}$ belongs to a compact subset of $C$; that is, the motion through $\varphi$ belongs to a compact subset of $C$.

From this lemma, we obtain

<u>Lemma 2.</u> If $\varphi$ in $C_H$ is such that the solution $x(\varphi)$ of system (1) with initial function $\varphi$ at $0$ is defined on $[-r, \infty)$ and $\|x_t(\varphi)\| \leq H_1 < H$ for $t$ in $[0, \infty)$, then $\Omega(\varphi)$ is a nonempty, compact, connected, invariant set and $\operatorname{dist}(x_t(\varphi), \Omega(\varphi)) \to 0$ as $t \to \infty$.

<u>Proof:</u> From Lemma 1, we know that the family of functions $x_t(\varphi)$, $t \geq r$, belongs to a compact subset $S$ of $C$ and, furthermore, $S$ could be chosen to be the set of $\psi$ in $C$ such that $\|\psi\| \leq H_1$, $\|\dot{\psi}\| \leq K$ for some constant $K$. This shows that $\Omega(\varphi)$ is nonempty and bounded.

If $\psi$ belongs to $\Omega(\varphi)$, then there exists a sequence $t_n$, $t_n \to \infty$ as $n \to \infty$, such that $\|x_{t_n}(\varphi) - \psi\| \to 0$ as $n \to \infty$. For any integer $N$, there exists a subsequence of the $t_n$, which we keep with the same designation, and a function $g_\tau(\varphi)$ defined ~~on $[-N, N]$~~ for $-N \le \tau \le N$ such that $\|x_{t_n+\tau}(\varphi) - g_\tau(\varphi)\| \to 0$ as $n \to \infty$, uniformly for $\tau$ in $[-N, N]$. By the diagonalization process, one can choose the $t_n$ so that $\|x_{t_n+\tau}(\varphi) - g_\tau(\varphi)\| \to 0$ as $n \to \infty$ uniformly on all compact subsets of $(-\infty, \infty)$. In particular, the sequence $x_{t_n+\tau}(\varphi)$ defines a function $g_\tau(\varphi)$ ~~on $(-\infty, \infty)$~~ for $-\infty < \tau < \infty$. It is easy to see that $g_\tau(\varphi)$ satisfies (1). Since $g_0(\varphi) = \psi$, it follows that the solution $x_t(\psi)$ of (1) with initial value $\psi$ at $0$ is defined for all values of $t$ in $(-\infty, \infty)$ and, furthermore, is in $\Omega(\varphi)$, since $\|x_{t_n+t}(\varphi) - x_t(\psi)\| \to 0$ as $n \to \infty$ for any fixed $t$. This shows that $\Omega(\varphi)$ is invariant. It is clear that $\Omega(\varphi)$ is connected.

To show $\Omega(\varphi)$ is closed, suppose $\psi_n$ in $\Omega(\varphi)$ approaches $\psi$ as $n \to \infty$. There exists an increasing sequence of $t_n = t_n(\psi_n) \to \infty$ as $n \to \infty$ such that $\|x_{t_n}(\varphi) - \psi_n\| \to 0$ as $n \to \infty$. Given any $\epsilon > 0$, choose $n$ so large that $\|\psi_n - \psi\| < \epsilon/2$ and $\|x_{t_n}(\varphi) - \psi_n\| < \epsilon/2$. Then $\|x_{t_n}(\varphi) - \psi\| < \epsilon$ for $n$ large enough which shows that $\psi$ is in $\Omega(\varphi)$ and $\Omega(\varphi)$ is closed. But, clearly $\Omega(\varphi) \subset S$ and since $S$ is compact, it follows that $\Omega(\varphi)$ is compact.

To show the last assertion of the lemma, suppose that there is an increasing sequence of $t_n \to \infty$ as $n \to \infty$ and an $\alpha > 0$ such that $\|x_{t_n}(\varphi) - \psi\| \geq \alpha$ for all $\psi$ in $\Omega(\varphi)$. Since $x_{t_n}(\varphi)$ belongs to a compact subset of $C$ there exists a subsequence which converges to an element $\psi$ in $C$ and thus $\psi$ is in $\Omega(\varphi)$. This is a contradiction of the above inequality and completes the proof of the lemma.

Remark. In the proof of ~~the~~ Lemma 2, we only used the fact that $x_t(\varphi)$ was continuous in $t, \varphi$ and that $x_t(\varphi)$ belonged to a compact subset of $C$. Therefore, the Lipschitz condition of $f$ could have been replaced by $\|\varphi\| \leq H_1 < H$ implies $|f(\varphi)| \leq L$ for some $L$.

Theorem 1. Let V be a continuous scalar function on $C_H$. If $U_\ell$ designates the region where $V(\varphi) < \ell$, suppose there exists a nonnegative constant K such that $|\varphi(0)| \leq K$, $V(\varphi) \geq 0$, and $\dot{V}_{(1)}(\varphi) \leq 0$ for all $\varphi$ in $U_\ell$. If R is the set of all points in $U_\ell$ where $\dot{V}_{(1)}(\varphi) = 0$ and M is the largest invariant set in R, then every solution of (1) with initial value in $U_\ell$ approaches M as $t \to \infty$.

The conditions $|\varphi(0)| \leq K$, $V(\varphi) \geq 0$ of this theorem can be replaced by the condition that the region where $V(\varphi) < \ell$ is compact, but the theorem as stated is more convenient in the applications. Theorem 1 together with Theorem 3 below generalize results of LaSalle for ordinary differential equations and the proofs are natural extensions of the ones given by LaSalle.

Proof: The conditions on V imply that $V(x_t(\varphi))$ is a nonincreasing function of t and $V(x_t(\varphi))$ is bounded below within $U_\ell$. Hence $\varphi$ in $U_\ell$ implies $x_t(\varphi)$ in $U_\ell$ and $|x(\varphi)(t)| \leq K$ for all $t \geq 0$ which implies $\|x_t(\varphi)\| \leq K$ for all $t \geq 0$; that is, $x_t(\varphi)$ is bounded and Lemma 2 implies $\Omega(\varphi)$ is an invariant set. But $V(x_t(\varphi))$ has a limit $\ell_0 < \ell$ as $t \to \infty$ and $V = \ell_0$ on $\Omega(\varphi)$. Hence $\Omega(\varphi)$ is in $U_\ell$ and $\dot{V}_{(1)} = 0$ on $\Omega(\varphi)$. Consequently, $\Omega(\varphi)$ invariant implies $\Omega(\varphi)$ is in M and Lemma 2 implies $x_t(\varphi) \to M$ as $t \to \infty$, completing the proof of the theorem.

Corollary 1. If the conditions of Theorem 1 are satisfied and $\dot{V}_{(1)}(\varphi) < 0$ for all $\varphi \neq 0$ in $U_\ell$, then every solution of (1) with initial value in $U_\ell$ approaches 0 as $t \to \infty$.

Notice that the conditions of Corollary 1 imply $f(0) = 0$ and 0 is in $U_\ell$.

Theorem 2. Suppose $f(0) = 0$ and there exists a function $u(s)$ which is continuous and increasing for $0 \leq s < H$ with $u(0) = 0$. In addition, assume there is a continuous scalar function $V(\varphi)$, $V(0) = 0$, defined on $C_H$ such that

$$u(|\varphi(0)|) \leq V(\varphi) \tag{3}$$

$$(4) \qquad \dot{V}_{(1)}(\varphi) \leqq 0,$$

for all $\varphi$ in $C_H$. Under these conditions, the solution $x = 0$ of (1) is stable. Furthermore, the solution $x = 0$ of (1) is asymptotically stable and every solution of (1) approaches $0$ as $t \to \infty$, provided the initial value $\varphi$ satisfies $V(\varphi) < \ell_o$, $\ell_o = \lim_{r \to H} u(r)$, and the only invariant set in $\dot{V}_{(1)}(\varphi) = 0$ is $0$.

<u>Proof.</u> There exists a function $w(s)$ continuous and nondecreasing for $s \geqq 0$ sufficiently small, $w(0) = 0$, and $V(\varphi) \leqq w(\|\varphi\|)$ for $\|\varphi\|$ sufficiently small. For any $\epsilon$, $0 < \epsilon < H$, choose $\delta < \epsilon$ so small that $w(\delta) < u(\epsilon)$. If $\varphi$ is in $C_\delta$, then $V(x_t(\varphi))$ nonincreasing implies

$$u(|x(\varphi)(t)|) \leq V(x_t(\varphi)) \leq V(\varphi) \leq w(\delta) < u(\epsilon)$$

for all $t \geq 0$. Therefore $|x(\varphi)(t)| < \epsilon$ for all $t \geq 0$ and, thus, $\|x_t(\varphi)\| < \epsilon$ for $t \geq 0$. This shows the solution $x = 0$ is stable. The second part of the theorem is proved as follows. Since $u$ is increasing the set $U_\ell$ of $\varphi$ for which $V(\varphi) < \ell$ satisfies the conditions of Theorem 1 if $\ell < \ell_o$ and, thus, by Corollary 1, every solution in $U_\ell$ approaches zero as $t \to \infty$. This completes the proof.

We can now deduce the following interesting corollary of Theorem 2. This corollary is also true for nonautonomous equations and may be found in Krasovskii [1]. Of course, the proof for the nonautonomous case is more difficult.

<u>Corollary 2.</u> Suppose $f(0) = 0$, $V(\varphi)$ satisfies condition (3) and there exists a function $w(s)$, $w(s)$ continuous, nonnegative and nondecreasing on $[0, H)$ such that

$$\dot{V}_{(1)}(\varphi) \leq - w(|\varphi(0)|).$$

Then the solution $x = 0$ of (1) is stable and if $w(s) < 0$ for $s \neq 0$, then it is asymptotically stable.

Proof: The stability follows immediately from Theorem 2. If $w(s) > 0$ for $s \neq 0$, then the largest invariant set in the set where $\dot{V}_{(1)} = 0$ must be those solutions of (1) for which $|x(t)| = 0$ for $-\infty < t < \infty$; that is the solution $x = 0$. This completes the proof.

Theorem 3. Let $C_H = C$ and $V$ be a continuous scalar function on $C$. If $V(\varphi) \geq 0$, $\dot{V}_{(1)}(\varphi) \leq 0$ for all $\varphi$ in $C$ and $R$ is the set of $\varphi$ in $C$ for which $\dot{V}_{(1)} = 0$ and $M$ is the larges invariant set in $R$, then all ~~bounded~~ solutions of (1) bounded for $t \geq 0$ approach $M$ as $t \to \infty$.

If, in addition, there exists a function $u(s)$, nonnegative and continuous for $0 \leq s < \infty$, $u(s) \to \infty$ as $s \to \infty$, such that

$$u(|\varphi(0)|) \leq V(\varphi) \tag{5}$$

for all $\varphi$ in $C$, then all solutions of (1) are bounded for all $t \geq 0$.

Proof: The first part of the theorem proceeds essentially as in Theorem 1. The boundedness property proceeds as follows. For any $\varphi_o$ in $C$, there is a constant $m$ such that $V(\varphi) > V(\varphi_o)$ for $|\varphi(0)| \geq m$. Since $V(x_t(\varphi))$ is a nonincreasing function of $t$, it follows that $|x(\varphi)(t)| < m$ for all $t \geq 0$ which implies $\|x_t(\varphi)\| < m$ for all $t \geq 0$ and the theorem is proved.

Corollary 3. If $f(0) = 0$, all of the conditions of Theorem 3 are satisfied and $V(0) = 0$, $\dot{V}_{(1)} < 0$ for $\varphi \neq 0$, then all solutions of (1) approach zero as $t \to \infty$ and the origin is globally asymptotically stable.

We next give a theorem on instability of the solution $x = 0$ of (1). In the statement of this result

$$\dot{V}^*_{(1)}(\varphi) = \overline{\lim_{h\to 0^+}} \frac{1}{h}\left[V(x_h(\varphi)) - V(\varphi)\right].$$

**Theorem 4.** Suppose $V(\varphi)$ is a continuous bounded scalar function on $C_H$ and there exist a $\gamma$ and an open set $U$ in $C$ such that the following conditions are satisfied:

i) $V(\varphi) > 0$ on $U$, $V(\varphi) = 0$ on that part of the boundary of $U$ in $C_\gamma$;

ii) $0$ belongs to the closure of $U \cap C_\gamma$;

iii) $V(\varphi) \leq u(|\varphi(0)|)$ on $U \cap C_\gamma$; $u(s)$, continuous, nonnegative and nondecreasing on $[0,H)$, $u(0)=0$;

iv) $\dot{V}^*_{(1)}(\varphi) \geq 0$ on $Cl(U \cap C_\gamma)$ and the set $R$ of $\varphi$ in $Cl(U \cap C_\gamma)$ such that $\dot{V}^*_{(1)}(\varphi) = 0$ contains no invariant set of (1) except $\varphi = 0$.

Under these conditions, the solution $x = 0$ of (1) is unstable and the trajectory of each solution of (1) with initial value in $U \cap C_\gamma$ must intersect $\bar{C}_\gamma$ in some finite time.

**Proof.** Suppose $\varphi_0 \in U \cap C_\gamma$. By hypothesis iii), $|\varphi_0(0)| \geq u^{-1}(V(\varphi_0))$ and iii) and iv) imply that $x_t = x_t(\varphi_0)$ satisfies

$$|x(t)| \geq u^{-1}(V(x_t)) \geq u^{-1}(V(\varphi_0))$$

as long as $x_t \in U \cap C_\gamma$. If $x_t$ leaves $U \cap C_\gamma$, then it must cross the boundary, $\partial C_\gamma$, of $C_\gamma$. In fact, it must cross either $\partial U$ or $\partial C_\gamma$, but it cannot cross $\partial U$ inside $C_\gamma$ since $V = 0$ on (that part of $\partial U$ inside $C_\gamma$) and $V(x_t) \geq V(\varphi_0) > 0$, $t \geq 0$. Now, suppose that $x_t$ never reaches $\partial C_\gamma$. Then $x_t$ belongs to a compact subset of $Cl(U \cap C_\gamma)$ for $t \geq 0$. Consequently, $x_t$ approaches $\Omega(\varphi_0)$, the $\omega$-limit set of $\varphi_0$, and $\Omega(\varphi_0) \subset Cl(U \cap C_\gamma)$. Since $V(x_t)$ is monotone nondecreasing and bounded above, it follows that $V(x_t) \to c$, a constant, as $t \to \infty$, and, thus, $\dot{V}_{(1)}(x_t(\psi)) = 0$ for $\psi$ in $\Omega(\varphi_0)$. Since $\psi$ in $\Omega(\varphi_0)$ implies $|\psi(0)| \geq u^{-1}(V(\varphi_0)) > 0$, this contradicts hypothesis (iv). Consequently, there is a value of $t_1$ such that $|x(t_1)| = \gamma$. Hypothesis ii) implies instability since $\varphi_0$ can be chosen arbitrarily close to zero. This completes the proof of the theorem.

<u>Remark.</u> Condition (iv) of Theorem 4 is certainly satisfied if there exists a continuous function $w(s)$ $0 \leq s < H$, increasing and positive for $s > 0$ such that

$$\dot{V}^{*}_{(1)}(\varphi) \geq w(|\varphi(0)|).$$

In fact, the largest invariant set in $U$ satisfying $\dot{V}^{*}_{(1)} = 0$ is obviously empty. Theorem 4 with condition (iv) replaced by this type of inequality also holds for nonautonomous systems. The reader can easily supply the proof.

3. <u>Applications.</u> The remainder of this paper is devoted to examples illustrating the application of these results to specific equations.

<u>Example 1.</u> Suppose $n = 1$ and

$$f(\varphi) = -\int_{-r}^{0} a(-\theta)g(\varphi(\theta))d\theta,$$

where $g(x)$ is a real function defined for all real $x$, locally Lipschitzian in $x$, ~~and there exists a constant M such that~~

(6) $$G(x) \overset{def}{=} \int_{0}^{x} g(s)ds \cancel{\geq M \text{ for all } x \text{ and } G(x)} \to \infty \quad \text{as} \quad |x| \to \infty,$$

and $a(t)$ is continuous together with its first and second derivatives on $[0, r]$. We also suppose that

(7) $$a(r) = 0,\ a(t) \geq 0,\ \dot{a}(t) \leq 0,\ \ddot{a}(t) \geq 0,\ 0 \leq t \leq r.$$

System (1) is then given by

(8) $$\dot{x}(t) = -\int_{-r}^{0} a(-\theta)g(x(t+\theta))d\theta = -\int_{t-r}^{t} a(t-u)g(x(u))du.$$

We wish to investigate along the same lines as Levin and Nohel [4] the relations between the solutions of (8) and the second order ordinary differential equation

(9) $$\ddot{x} + a(0)g(x) = 0.$$

First of all, we derive the second order functional-differential equation ~~for~~ which all solutions of (8) must satisfy. If $x$ is a solution of (8), then it has continuous second derivatives. Differentiation of (8) and making use of (7) yields

$$\ddot{x}(t) + a(0)g(x(t)) = - \int_{t-r}^{t} \dot{a}(t - u)g(x(u))du$$

$$= - \dot{a}(t - \theta) \int_{t}^{\theta} g(x(u))du \Big|_{t-r}^{t} \quad \int_{t-r}^{t} \ddot{a}(t - \theta) \left(\int_{t}^{\theta} g(x(u))du\right)d\theta$$

$$= \dot{a}(r) \int_{t}^{t-r} g(x(u))du - \int_{t-r}^{t} \ddot{a}(t - \theta)\left(\int_{t}^{\theta} g(x(u))du\right)d\theta$$

$$= - \dot{a}(r) \int_{-r}^{0} g(x(t + \theta))d\theta + \int_{-r}^{0} \ddot{a}(-\theta)\left( \int_{\theta}^{0} g(x(t + u))du\right)d\theta.$$

Consequently, every solution of (8) must satisfy the equation

(10) $$\ddot{x}(t) + a(0)g(x(t)) = - \dot{a}(r) \int_{-r}^{0} g(x(t + \theta))d\theta + \int_{-r}^{0} \ddot{a}(-\theta)\left(\int_{\theta}^{0} g(x(t + u))du\right)d\theta$$

for $t > 0$.

Equation (10) arises in the applications in the problem of the stability of a circulating fuel nuclear reactor (see W. K. Ergen [5]. In this case $x$ represents the neutron density in the reactor. Also, the same equation arises in some one dimensional problems in viscoelasticity where $x$ then represents the strain and $a$ the relaxation function.

Suppose $y(t)$ is any continuous function defined on $(-\infty, \infty)$. In the following, we shall be interested in such functions $y(t)$ which satisfy the additional conditions:

(11 - a) if $\dot{a}(r) \neq 0$, then $\int_{-r}^{0} g(y(t + \theta))d\theta = 0$; for $-\infty < t < \infty$

(11 - b) if $\dot{a}(s) \not\equiv 0$ for some $s$ in $[0, r]$, then $\int_{-s}^{0} g(y(t + \theta))d\theta = 0$ for $-\infty < t < \infty$.

We shall say that <u>$\Omega(\varphi)$, the $\omega$-limit set of an orbit of (8) thru $\varphi$, is the union of orbits of solutions of (9) which satisfy (11)</u> if $\Omega(\varphi)$ is the union of sets of the form $\bigcup_{-\infty<t<\infty} u_t(\alpha, \beta)$, where $u(\alpha, \beta)(t)$ is a solution of (9) defined for $-\infty < t < \infty$, $u(\alpha, \beta)(0) = \alpha$, $\dot{u}(\alpha, \beta)(0) = \beta$ and $u(\alpha, \beta)(t)$ satisfies (11).

Notice that any solution of (9) which satisfies (11) also satisfies (10).

Not all solutions of (10) are solutions of (8). In fact, integration of (10) from 0 to $t$ yields

$$(12) \qquad \dot{x}(t) - \dot{x}(0+) = -\int_{t-r}^{t} a(t - u)g(x(u))du + \int_{-r}^{0} a(-\theta)g(x(u))du,$$

which is equivalent to (10). Given any initial function $\varphi$ in $C$, one must choose $\dot{x}(0+)$ in a special manner to obtain a solution of (8). In fact, $\dot{x}(0+)$ must be such that

$$\dot{x}(0+) = -\int_{-r}^{0} a(-\theta)g(\varphi(\theta))d\theta .$$

This is an additional restriction that must be satisfied by the solutions of the second order ordinary differential equation (9).

If we define a solution of (10) to be a function which is continuous together with its first derivative, then a solution $x$ with initial function $\varphi$ at $t = 0$ is such that $\dot{x}(0+) = \dot{\varphi}(0^-)$ and any solution of (10) is a solution of the system

$$(13) \qquad \dot{x}(t) = -\int_{-r}^{0} a(-\theta)g(x(t + \theta))d\theta + C$$

where $C$ is given by

$$C = \dot{\varphi}(0) + \int_{-r}^{0} a(-\theta)g(\varphi(\theta))d\theta$$

and $\varphi$ is the initial function for $x$. Therefore, any statement about the solutions of (8) is a statement about the solutions of (10) provided the initial function satisfies conditions which ensure that $C = 0$. Notice that any constant function is a solution of (10) and this is not the case for (8). Later, we will discuss some more specific relations between (8) and (10) under more restrictive conditions on the function $g$.

Theorem 5. If system (8) satisfies conditions (6) and (7), then every solution of (8) is bounded and the $\omega$-limit set of any solution of (8) is the union, as explained above, of orbits of solutions of (9) which satisfy (11).

Proof: We use the same Lyapunov functional introduced by Levin and Nohel [4]. If $M$ is the minimum of $G(x)$ defined in (6) and

$$V(\varphi) = G(\varphi(0)) - \frac{1}{2}\int_{-r}^{0} \dot{a}(-\theta)[\int_{\theta}^{0} g(\varphi(s))ds]^2 d\theta - M,$$

then

$$\dot{V}_{(8)}(\varphi) = \frac{1}{2}\dot{a}(r)[\int_{-r}^{0} g(\varphi(\theta))d\theta]^2 - \frac{1}{2}\int_{-r}^{0} \ddot{a}(-\theta)[\int_{\theta}^{0} g(\varphi(s))ds]^2 d\theta \leq 0$$

by the hypothesis (7). The hypotheses also imply that the conditions of Theorem 3 are satisfied and, thus, every solution of (8) is bounded and must approach the largest invariant set of (8) in the set where $\dot{V}_{(8)} = 0$. It is clear that this latter set $R$ consists of all those $\varphi$ in $C$ for which

$$\int_{-r}^{0} g(\varphi(\theta))d\theta = 0 \quad \text{if} \quad \dot{a}(r) \neq 0$$

(14)

$$\int_{-s}^{0} g(\varphi(\theta))d\theta = 0 \quad \text{for any } s \text{ in } [0, r] \text{ for which } \ddot{a}(s) \neq 0.$$

But all solutions of (8) satisfy (10) and any solution in an $\omega$-limit set must be defined on $(-\infty, \infty)$ and satisfy (14) for all $t$ in $(-\infty, \infty)$. Finally this implies these solutions must satisfy (9) amd (11), completing the proof of the theorem.

Two questions now present themselves in a natural manner: i) What are the possible solutions of (9) which satisfy (11)? What additional conditions on $a(t)$ and $g(x)$ will ensure that $\Omega(\varphi)$ is generated by only one solution of (9)? We now give some partial results in an attempt to answer these questions.

Corollary 4. If system (8) satisfies conditions (6) and (7) and there is an $s$ in $[0, r]$ such that $\ddot{a}(s) > 0$, then (8) has no nonconstant periodic solutions.

Proof: There exist $s_o < s_1$ such that $\ddot{a}(s) > 0$ for $s_o \leq s \leq s_1$. If (8) has a periodic solution $x_t(\varphi)$ then $\Omega(\varphi) = U_t x_t(\varphi)$, and Theorem 5 states that it must be generated by a nonconstant periodic solution $u(t)$ of least period $p$ of (9) satisfying (11 - b) for $s_o < s < s_1$. But integration of (9) yields

$$\dot{u}(t) - \dot{u}(t - s) = -a(0) \int_{-s}^{0} g(u(t + \theta))d\theta = 0,$$

which implies $\dot{u}(t)$ is periodic of period $s$ for $s_o < s < s_1$. But this is impossible since $\dot{u}(t)$ is periodic of period $p$. This proves the corollary.

Corollary 5. If system (8) satisfies conditions (6), (7), $xg(x) > 0$ for $x \neq 0$ and there is an $s$ in $[0, r]$ such that $\ddot{a}(s) > 0$, then the solution $x = 0$ of (8) is globally asymptotically stable.

Proof: The condition $\ddot{a}(s) > 0$ for some $s$ implies $a(0) > 0$ and this together with the conditions on $g(x)$ imply that every solution of (9) is periodic and the only constant solution is zero. Theorem 5 and Corollary 2 imply the conclusions of the theorem.

Corollary 5 is essentially the same as the one of Levin and Nohel [4].

Corollary 5 is a statement of global asymptotic stability of the solution $x = 0$ of (8). Can we use this result to draw any conclusions about the solutions of (10)? Any constant function satisfies (10) and one would suspect that any solution of (10) approaches a constant as $t \to \infty$. This seems rather difficult to prove in the general case, but we can prove the following simple result from Corollary 5. <u>If for any real constant $b$, there exists a real $\delta$ such that</u>

$$h(y) = g(y + \delta) - b$$

<u>satisfies the condition</u>

$$yh(y) > 0$$

<u>where $g(y)$ satisfies the conditions of corollary 4, then for a given solution $x$ of (10), there is a constant $a$ such that $x \to a$ as $t \to \infty$.</u> In fact, if the solutions of (10) are required to have continuous first derivatives as stated in the remarks preceding Theorem 5, then the system (10) is equivalent to the system

$$\dot{x}(t) = -\int_{-r}^{0} a(-\theta)[g(x(t+\theta)) - C/\gamma]d\theta$$

$$C = \dot{\varphi}(0^-) + \int_{-r}^{0} a(-\theta)g(\varphi(\theta)), \quad \gamma = \int_{-r}^{0} a(-\theta)d\theta$$

and $\varphi$ is the initial value of the solution $x$ under investigation. If we let $b = C/\delta$, $\delta$ be defined as above and $x = y + \delta$, then $y$ satisfies (8) with some appropriate initial values and approaches zero by Corollary 5. Thus, we have proved the above result.

There are many functions $g(x)$ which will be such that $h(y)$ satisfies the above conditions. In fact, if $xg(x) > 0$ for $x \neq 0$, $|g(x)| \to \infty$ with $|x|$, $g'(x) > 0$ for all $x$, then this is true. In particular, the functions $g(x) = e^x - 1$, $g(x) = x$, etc. satisfy these conditions.

Now, let us consider a more particular case and investigate the possible $\omega$-limit sets of (8). Let $a(0) = 1$ and

$$g(x) = -x + x^3. \tag{15}$$

Suppose there are $s_0 < s_1$ such that $\ddot{a}(s) > 0$, $s_0 \leq s \leq s_1$. The conditions of Theorem 5 and Corollary 4 are satisfied and the $\omega$-limit set of any solution of (8) with $g(x)$ given by (15) must satisfy

$$\begin{aligned} &\ddot{x} - x + x^3 = 0 \\ &\int_{-s}^{0} g(x(t+\theta))d\theta = 0 \quad \text{for all } s \text{ in } [s_0, s_1] \end{aligned} \tag{16}$$

The phase portrait of the solutions of (16) is easily shown to be as in Fig. 1. By Corollary 4, the only possible candidates for an $\Omega(\varphi)$ of (8) are shown in Fig. 2. Now the curves $C_1$, $C_2$ are determined by solutions $u_1(t) > 0$, $u_2(t) < 0$, $-\infty < t < \infty$, respectively. But there is a $T$ such that $0 < u_1(t) < 1$, $-1 < u_2(t) < 0$, for $t \geq T$ and, thus, for any $s \neq 0$

$$\int_{-s}^{0} g(u_1(t+\theta))d\theta < 0, \quad \int_{-s}^{0} g(u_2(t+\theta))d\theta > 0 \quad \text{for } t \geq T.$$

This shows that $u_1(t)$, $u_2(t)$ cannot have orbits which belong to an $\Omega(\varphi)$. Consequently, the $\omega$-limit set of any solution of (8) with $g$ satisfying (15) must be one of the three constant functions $-1, 0, 1$.

For this same particular example let us analyze the stability properties of each of the constant solutions $-1, 0, 1$. If $\alpha$ is either of these solutions,

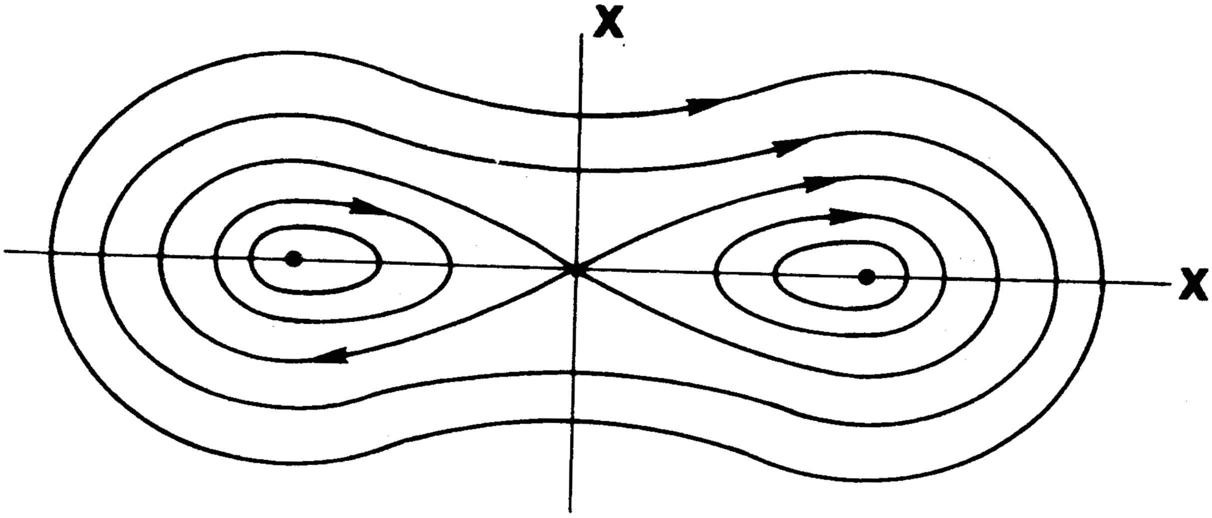


Fig. 1

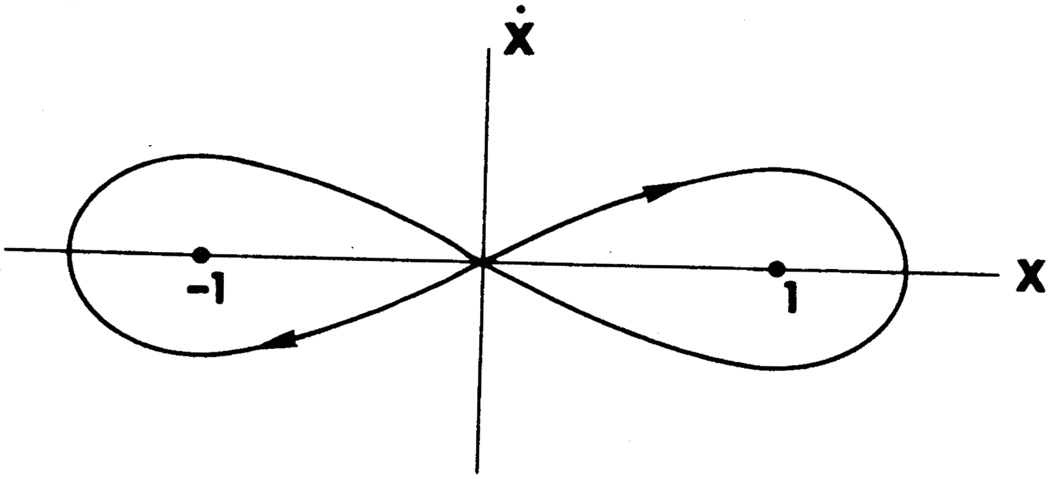


Fig. 2

the linear variational equations relative to $\alpha$ are

$$(17) \qquad \dot{y}(t) = -\int_{t-r}^{t} a(t-\tau)(-1+3\alpha^2)y(\tau)d\tau .$$

If the solution $y = 0$ is asymptotically stable or unstable then it follows from results of stability theory that the same is true for the complete variational equations. For $\alpha = \pm 1$, Corollary 5 implies the solution $y = 0$ of (17) is asymptotically stable and using the negative of the V-functional employed in the proof of Theorem 5 and applying Theorem 4, one finds that for $\alpha = 0$, the solution $y = 0$ of (17) is unstable. Consequently, any nonconstant solution of (8) with $g$ satisfying (15) approaches either $+1$ or $-1$ as $t \to \infty$.

It should be clear the particular form $g(x)$ given in (15) is not at all essential for the above type of results and one could have a function $g(x)$ with many zeros on the axis.

If $\ddot{a}(s) = 0$ for all $s$ then the behavior of solutions of (8) may be more complicated. The following result indicating this fact is due to Levin and Nohel [4]. The proof of the theorem is essentially the same as in [4].

<u>Theorem 6.</u> If system (8) satisfies (6), (7), and $a(s) = (r-s)/r$, $0 \leq s \leq r$, $xg(x) > 0$ for $x \neq 0$, then every solution of (8) is bounded and the $\omega$-limit set of any solution of (8) consists of the orbit of some solution $u(t)$ of (9) which satisfies the condition

$$(18) \qquad \int_{-r}^{0} g(u(t+\theta))d\theta = 0, \; -\infty < t < \infty.$$

In this case, $u(t)$ must be periodic of some period $p$ and there must be an integer $m$ such that $mp = r$. If $\ddot{u} + g(u) = 0$ and $mp = r$ then $u$ satisfies (8).

<u>Remark.</u> If the only solution of (9) which satisfies (18) is the solution $x(t) = 0$ for all $t$, the zero solution of (8) is globally asymptotically stable. In particular, this is true if $g(x) = \sigma^2 x$, $\sigma^2 > 0$, and $r \not\equiv 0 \pmod{2\pi/\sigma}$. If $r \equiv 0 \pmod{2\pi/\sigma}$ and $g(x) = \sigma^2 x$, then every periodic solution of (9) is a periodic solution of (8). Also, it will be clear from the proof that $g(0) = 0$ and (6) implies the zero solution of (8) is uniformly stable.

<u>Proof of Theorem 6.</u> From Theorem 5, $\Omega(\varphi)$ must be the union of orbits of solutions of (9) which satisfy (18) since $\dot{a}(s) = 0$ for all $s$. All solutions of (9) are periodic and it is easy to see that (18) implies the periods must satisfy the property stated in the theorem. We now analyze the detailed structure of $\Omega(\varphi)$. Let $\Omega(\varphi)$ be the $\omega$-limit set of an element $\varphi$ in $C$ obtained by moving along trajectories of (8). Choose $V$ as the function used in the proof of Theorem 5 except with $M = 0$. Since $\Omega(\varphi)$ is invariant, $V(x_t(\psi))$ is constant for $\psi \in \Omega(\varphi)$, $\infty < t < \infty$. Therefore, $V(x_t(\psi)) = 0$ for $\psi \in \Omega(\varphi)$ if and only if $\Omega(\varphi) = \{0\}$. Now suppose $0 \notin \Omega(\varphi)$. Then $V(x_t(\psi)) = \nu > 0$ for $\psi \in \Omega(\varphi)$, $-\infty < t < \infty$. Furthermore, $\Omega(\varphi)$ consists of periodic solutions (9) which satisfy (18) and the only periodic solutions of (9) which satisfy (18) must have a period $p(\alpha)$ such that $m(\alpha)p(\alpha) = r$ for some integer $m(\alpha)$. On the other hand, since $\Omega(\varphi)$ is connected and $p(\alpha)$ is continuous, this implies all solutions of (9) which lie in $\Omega(\varphi)$ must have the same period; that is, $mp = r$, $m$ and $p$ independent of $\alpha$.

Now let $u(t, \alpha)$, $u(0, \alpha) = -\alpha < 0$, $\dot{u}(0, \alpha) = 0$, $0 < \alpha_1 \leq \alpha \leq \alpha_2$ be those periodic solutions of (9) which lie in $\Omega(\varphi)$, $0 \notin \Omega(\varphi)$. Then each $u(t, \alpha)$, $\alpha_1 \leq \alpha \leq \alpha_2$ has the same period $p$ with $mp = r$ for some integer $m$ and $V(u_t(\alpha)) = \nu > 0$, $\alpha_1 \leq \alpha \leq \alpha_2$, $-\infty < t < \infty$. If we show that $\alpha_1 = \alpha_2$, this will complete the proof of the theorem. Using the fact that $\ddot{u}(t, \alpha) = -g(u(t, \alpha))$, we see that

$$V(u_t(\alpha)) = G(u(t, \alpha)) + \frac{1}{2}\dot{u}^2(t, \alpha) + \frac{1}{2r}\int_{-r}^{0} \dot{u}^2(t + \tau, \alpha)d\tau .$$

Since

$$G(u(t, \alpha)) + \frac{1}{2}\dot{u}^2(t, \alpha) = G(-\alpha)$$

for all $t$, this implies

$$V(u_t(\alpha)) = G(-\alpha) + \frac{1}{2r}\int_{-r}^{0} \dot{u}^2(t + \tau, \alpha)d\tau$$

$$= G(-\alpha) + \frac{1}{2p}\int_{-p}^{0} \dot{u}^2(\tau, \alpha)d\tau.$$

The latter relation follows because $\dot{u}$ is periodic of period $p = r/m$. Furthermore, from the symmetry of the curves defined by $u(t, \alpha)$, $\dot{u}(t, \alpha)$ in the $u, \dot{u}$ plane, it follows that

$$V(u_t(\alpha)) = G(-\alpha) + \frac{1}{2p}\int_{-p}^{0} [G(-\alpha) - G(u(\tau, \alpha))]d\tau$$

$$= G(-\alpha) + \frac{\sqrt{2}}{p}\int_{-\alpha}^{\beta(\alpha)} [G(-\alpha) - G(\xi)]^{\frac{1}{2}}d\xi$$

where $\beta(\alpha) > 0$ is defined as the unique positive number such that $G(-\alpha) = G(\beta(\alpha))$. From the properties of $g$, it follows that $V(u_t(\alpha))$ is a strictly increasing function of $\alpha$ and this is a contradiction unless $\alpha_1 = \alpha_2$.

To prove the last assertion of the theorem, note that

$$-\int_{t-r}^{t} \frac{L - (t - u)}{L} g(x(u))du = \int_{t-r}^{t} \frac{L - (t - u)}{L} \ddot{u}\, du = \dot{u}(t)$$

from an integration by parts and the periodicity. This completes the proof of Theorem 6.

Can we use this result to draw some conclusions about system (10)? Let us suppose $g(x)$ is such that $|g(x)| \to \infty$ with $|x|$ and for any real constant $b$ there is a $\delta$ such that

$$h(y) \overset{\text{def}}{=} g(y + \delta) - b$$

satisfies $yh(y) > 0$ for $y \neq 0$, then using the same argument as that after Corollary 4, the transformation $x = y + \delta$ where $g(\delta) = C/\gamma$, $\gamma = \int_{-r}^{0} a(\theta)d\theta$, $C = \dot{\varphi}(0) + \int_{-r}^{0} a(-\theta)y(\theta)d\theta$, yields the equation

$$\dot{y}(t) = -\int_{-r}^{0} a(-\theta)h(y(t + \theta))d\theta.$$

The function $h$ satisfies the conditions of Theorem 6 and therefore, the $\omega$-limit set of any solution consists of an orbit of

$$\ddot{y} + h(y) = 0$$

generated by a periodic solution of period $p$ with $mp = r$ for some integer $m$. Therefore, every solution of (10) is asymptotic to a constant plus a periodic solution for large values of $t$.

Using the same $V$ functional as in the proof of Theorem 5 and assuming that $g(x)$, $a(t)$ satisfy the conditions (6), (7), and in addition $F(x)$ is any continuous function of $x$ which is locally Lipschitzian in $x$, and satisfies

$$g(x)F(x) > 0 \quad \text{for} \quad x \neq 0, \quad F(0) = 0, \quad g(0) = 0,$$

one shows that the zero solution of the equation

$$\dot{x}(t) = F(x(t)) - \int_{t-r}^{t} a(t - \tau)g(x(u))du$$

is globally asymptotically stable.

Example 2. (Volterra, [6].) Consider the equation

$$A\dot{x}(t) + Bx(t) = \int_0^r F(\theta)x(t-\theta)d\theta \tag{19}$$

where $A, B, F$ are symmetric matrices, $F$ is continuously differentiable and

$$A > 0,\ F(\theta) \geq 0,\ \dot{F}(\theta) \leq 0,\ 0 \leq \theta \leq r,\ M \stackrel{\text{def}}{=} B - \int_0^r F(\theta)d\theta \geq 0. \tag{20}$$

With $M$ defined as above and $\dot{x}(t) = y(t)$, we can rewrite (19) as

$$\begin{aligned} \dot{x}(t) &= y(t) \\ A\dot{y}(t) &= -Mx(t) + \int_0^r F(\theta)[x(t-\theta) - x(t)]\,d\theta \end{aligned} \tag{21}$$

System (21) is more general than (19) and for (21) to be equivalent to (19) the initial conditions for (21) must be chosen in a special manner. However, we are going to discuss the more general system (21). Following Volterra [6], we define $V(\varphi, \psi)$ as

$$V(\varphi,\psi) = \frac{1}{2}\varphi'(0)M\varphi(0) + \frac{1}{2}\psi'(0)A\psi(0) + \frac{1}{2}\int_0^r[\varphi(-\theta)-\varphi(0)]'F(\theta)[\varphi(-\theta)-\varphi(0)]d\theta. \tag{22}$$

Then

$$\begin{aligned} \dot{V}_{(21)}(x_t,y_t) &= y'(t)Mx(t)-y'(t)Mx(t) + y'(t)\int_0^r F(\theta)[x(t-\theta)-x(t)]d\theta \\ &+ \frac{d}{dt}\frac{1}{2}\int_{t-r}^{t}[x(u)-x(t)]'F(t-u)[x(u)-x(t)]du \\ &= y'(t)\int_0^r F(\theta)[x(t-\theta)-x(t)]d\theta - \frac{1}{2}[x(t-r) - x(t)]'F(r)[x(t-r) - x(t)] \\ &+ \frac{1}{2}\int_{t-r}^{t}[x(u)-x(t)]'\dot{F}(t-u)[x(u)-x(t)]du - \int_{t-r}^{t} y'(t)F(t-u)[x(u)-x(t)]du \\ &= -\frac{1}{2}[x(t-r)-x(t)]'F(r)[x(t-r)-x(t)] + \frac{1}{2}\int_0^r[x(t-\theta)-x(t)]'\dot{F}(\theta)[x(t-\theta)-x(t)]d\theta \end{aligned}$$

Consequently,

$$\dot{V}_{(21)}(\varphi,\psi) = -\tfrac{1}{2}[\varphi(-r)-\varphi(0)]' F(r)[\varphi(-r)-\varphi(0)] + \tfrac{1}{2}\int_0^r [\varphi(-\theta)-\varphi(0)]' \dot{F}(\theta)[\varphi(-\theta)-\varphi(0)]d\theta \leq 0. \tag{23}$$

<u>Theorem 7.</u> (Volterra [6]). If $A > 0$, $M > 0$, $F(\theta) \geq 0$, and there is a $\theta_0$ in $[0, r]$ such that $\dot{F}(\theta_0) < 0$, then every solution of (21) approaches zero as $t \to \infty$.

<u>Remark.</u> This theorem is actually a generalization of the result of Volterra, since he assumed $F(r) = 0$, $\dot{F}(\theta) < 0$ for all $\theta$ in $[0, r]$ and only showed that all solutions of (21) were bounded and there were no constant or periodic solutions. Of course, these conclusions are enough to imply that all solutions approach zero as $t \to \infty$.

<u>Proof of Theorem 7.</u> Since $\dot{F}(\theta_0) < 0$, there exist $\theta_1 < \theta_2$ such that $\dot{F}(\theta) < 0$ for $\theta_1 \leq \theta \leq \theta_2$. Furthermore, from (23), $\dot{V}_{(21)}(\varphi, \psi) = 0$ implies $\varphi(-\theta) = \varphi(0)$ for all $\theta$ for which $\dot{F}(\theta) < 0$; in particular, for $\theta_1 \leq \theta \leq \theta_2$. For a solution $x_t$, $y_t$ of (21) to belong to the largest invariant set in the set where $\dot{V} = 0$, we must have $x(t - \theta) = x(t)$ for all $t$ in $(-\infty, \infty)$ and $\theta_1 \leq \theta \leq \theta_2$. Consequently $x_t$ must be a constant function which implies $y_t = 0$, which in turn implies $Mx(t) = 0$ and $M > 0$ implies $x(t) = 0$ for all $t$. Thus, the largest invariant set of (21) in the set where $\dot{V}_{(21)}(\varphi, \psi) = 0$ is $(0, 0)$. Since $V$ satisfies the conditions of Theorem 2, the origin is asymptotically stable. This completes the proof of the theorem.

<u>Theorem 8.</u> If $A > 0$, $M > 0$, $\dot{F}(\theta) = 0$, $F > 0$, $0 \leq \theta \leq r$, then all solutions of (21) are bounded and the $\omega$-limit set of any solution of (21) must be

generated by periodic solutions of period $r$ of the system

$$\dot{x} = y$$
$$(24)$$
$$A\dot{y} = -Bx \quad .$$

Proof: Using relation (23) for $\dot{V}_{(21)}$ we see that the set where $\dot{V} = 0$ consists precisely of those initial functions for which $\varphi(-r) = \varphi(0)$. Consequently for a solution to belong to the set where $\dot{V} = 0$ it must be periodic of period $r$. But any nontrivial periodic solution of a linear equation with constant coefficients has zero average over the period and it follows that the largest invariant set in the set where $\dot{V} = 0$ must be periodic solutions of period $r$ which satisfy (24). The conditions of Theorem 2 are satisfied and the theorem is proved.

In the case where $x$ is a scalar, it is easy to show that the $\omega$-limit set of any solution of (21) satisfying the conditions of Theorem 8 is generated by at most one solution of (24).

We now prove the following instability theorem for the case where $M < 0$.

Theorem 9. If $A > 0$, $M < 0$, $F(\theta) > 0$, $0 \leq \theta < r$, $\dot{F}(\theta) \leq 0$, $0 \leq \theta \leq r$, and there is a $\theta_0$ in $[0, r]$ such that $\dot{F}(\theta_0) < 0$, then the solution $x = 0$, $y = 0$ of (21) is unstable.

Proof: If $V$ is chosen as in (22), then $\dot{V}^*_{(21)}$ satisfies (23). The same argument as given in the proof of Theorem 7 shows that the largest invariant set of (21) in the set where $\dot{V}^*_{(21)} = 0$ is $(0, 0)$. Furthermore, $V(\varphi, \psi) < 0$ if and only if

$$\frac{1}{2}\psi'(0)A\psi(0) + \frac{1}{2}\int_0^r [\varphi(-\theta) - \varphi(0)]' \; F(\theta)[\varphi(-\theta) - \varphi(0)]d\theta < - \frac{1}{2}\varphi'(0)M\varphi(0)$$

The set $U$ of $\varphi, \psi$ satisfying this relation is certainly open and $0$ belongs to the boundary of $U$. The conditions of Theorem 4 are satisfied for the function $-V$ and any solution of (21) with initial values $(\varphi, \psi) \neq (0, 0)$, $(\varphi, \psi)$ in $U$ must approach infinity, completing the proof of the theorem.

Example 3. (Interaction of species, Volterra [7]). In [7], Volterra has discussed the following model for the interaction of two species. Let $N_1(t)$, $N_2(t)$ be the numbers of species of type A, B respectively, at time $t$. Species A has an unlimited food supply and species B relies upon A for his development. For positive constants $r$, $\epsilon_j$, $\gamma_j$, $j = 1, 2$, and nonnegative functions $F_1(\theta)$, $F_2(\theta)$, $0 \leq \theta \leq r$, the species are assumed to evolve according to the relations

$$
(25) \qquad
\begin{aligned}
\dot{N}_1(t) &= [\epsilon_1 - \gamma_1 N_2(t) - \int_0^r F_1(\theta) N_2(t - \theta)] N_1(t), \\
\dot{N}_2(t) &= [-\epsilon_2 + \gamma_2 N_1(t) + \int_0^r F_2(\theta) N_1(t - \theta)] N_2(t).
\end{aligned}
$$

In the classical model of the interaction of A and B, the functions $F_1$, $F_2$ are zero. The additional terms involving $F_1(\theta)$, $F_2(\theta)$ allow for more interaction between the species A and B. For the simple model it is well known that all solutions of (25) with $N_1 > 0$, $N_2 > 0$ are periodic and encircle an equilibrium point in the $(N_1, N_2)$ - plane. As we will see below, the more general system (25) has an asymptotically stable equilibrium point under very general assumptions on $F_1$, $F_2$.

The equilibrium points of (25) are $(0, 0)$ and $(K_1, K_2)$, $K_1 = \epsilon_2/(\gamma_2 + \Gamma_2)$ $K_2 = \epsilon_1/(\gamma_1 + \Gamma_1)$, $\Gamma_j = \int_0^r F_j(\theta)d\theta$, $j = 1, 2$. The point $(0, 0)$ is obviously unstable. We wish now to analyze the behavior of the solutions of (25) in a neighborhood of the equilibrium point $(K_1, K_2)$. If we let $N_1 = K_1(1 + x)$, $N_2 = K_2 (1 + y)$, then we obtain the linear variational equations

$$\dot{x}(t) = -py(t) - \int_0^r G(\theta)[y(t-\theta) - y(t)]d\theta$$

(26)

$$\dot{y}(t) = qx(t) + \int_0^r F(\theta)[x(t-\theta) - x(t)]d\theta$$

where $p = \gamma_1 K_2 + \int_0^r G(\theta)d\theta$, $q = \gamma_2 K_1 + \int_0^r F(\theta)d\theta$, $G = K_2 F_1 \geq 0$, $F = K_1 F_2 \geq 0$, and we make the assumption that $p > 0$, $q > 0$. If we can show that system (26) is asymptotically stable, then the equilibrium point $(K_1, K_2)$ of (25) is asymptotically stable. Notice that (26) is a generalization of the scalar version of (21). In fact, we obtain (21) by putting $G = 0$ and replacing $y$ by $-y$.

<u>Theorem 10.</u> If $p > 0$, $q > 0$, $F(\theta) \geq 0$, $G(\theta) \geq 0$, $\dot{F}(\theta) \leq 0$, $\dot{G}(\theta) \leq 0$, $0 \leq \theta \leq r$, and there is a $\theta_0$ in $[0, r]$ such that either $\dot{F}(\theta_0) < 0$ or $\dot{G}(\theta_0) < 0$, then every solution of (26) approaches zero as $t \to \infty$.

<u>Proof:</u> Define

$$V(x_t, y_t) = \frac{1}{2}p\, y^2(t) + \frac{1}{2}qx^2(t) + \frac{1}{2}\int_0^r G(\theta)[y(t-\theta) - y(t)]^2 d\theta$$
$$+ \frac{1}{2}\int_0^r F(\theta)[x(t-\theta) - x(t)]^2 d\theta.$$

It follows that

$$\dot{V}_{(26)}(x_t, y_t) = \frac{1}{2}\int_0^r \dot{G}(\theta)[y(t-\theta) - y(t)]^2 d\theta$$
$$+ \frac{1}{2}\int_0^r \dot{F}(\theta)[x(t-\theta) - x(t)]^2 d\theta \leq 0.$$

Therefore all solutions of (26) are bounded. If there is a $\theta_0$ in $[0, r]$ such that $\dot{F}(\theta_0) < 0$, then $x_t$ in the largest invariant set in the set where $\dot{V} = 0$ must satisfy $x(t-\theta) - x(t) = 0$ for $\theta$ in an interval containing $\theta_0$. Consequently, $x = c$ a constant and the differential equation implies $x = 0$, $y = 0$.

Therefore, Theorem 2 implies every solution approaches zero as $t \to \infty$. The other case is treated in exactly the same manner and the theorem is proved.

One could obviously generalize this example to a system of equations of a special type.

Example 4. Let us consider a special nonlinear version of equation (19); namely, the second order system

$$\dot{x}(t) = y(t)$$
$$(27) \qquad \dot{y}(t) = -h(x(t)) + \int_0^h F(\theta)g[x(t-\theta) - x(t)]d\theta$$

where $h(x)$, $g(x)$ are continuous functions of $x$ such that $xh(x) > 0$, $xg(x) > 0$ if $x \neq 0$ and

$$H(x) \stackrel{\text{def}}{=} \int_0^x h(s)ds \to \infty \text{ as } |x| \to \infty.$$

If $G(x) = \int_0^x g(s)ds$ and

$$V(\varphi, \psi) = \frac{1}{2}\psi^2(0) + H(\varphi(0)) + \int_0^h F(\theta)G[\varphi(-\theta) - \varphi(0)]d\theta,$$

one easily obtains

$$\dot{V}_{(27)}(x_t, y_t) = -F(r)G[x(t-r) - x(t)] + \int_0^r \dot{F}(\theta)G[x(t-\theta) - x(t)]d\theta.$$

If we suppose $F(\theta) \geq 0$ for all $\theta$ and there is a $\theta_0$ such that $\dot{F}(\theta_0) < 0$, then, as in the previous example 2, one shows that every solution of (27) approaches zero as $t \to \infty$. If $\dot{F}(\theta) = 0$, $0 \leq \theta \leq r$ and $F(r) > 0$, then every solution of (27) is bounded and the $\omega$-limit set of any solution of (27) must consist of periodic solutions of

$$\dot{x} = y$$
$$(28) \qquad \dot{y} = -h(x)$$

of period $r$. If there are no periodic solutions of (28) of period $r$ except the solution $(0, 0)$, then every solution of (27) approaches zero.

Example 5. (Krasovskii, [1, p. 173]) Consider the equation

$$\dot{x}(t) = y(t)$$

(29)

$$\dot{y}(t) = -f(y(t)) - g(x(t)) + \int_{-r}^{0} h(x(t+\theta))y(t+\theta)d\theta$$

where

$$f(x)/x > b > 0, \quad g(x)/x > a > 0, \quad |h(x)| < L, \quad r < b/L,$$

for all $x$. If $h(x) = dg(x)/dx$, system (29) is related to a second order differential-difference equation. For any scalar functions $\varphi, \psi$ defined and continuous on $[-r, 0]$, define

$$V(\varphi, \psi) = 2G(\varphi(0)) + \psi^2(0) + \nu\int_{-r}^{0}\left(\int_{\tau}^{0}\psi^2(\theta)d\theta\right)d\tau$$

where $G(x) = \int_0^x g(s)ds$ and $\nu$ is a positive constant to be determined.

A direct computation shows that

$$\dot{V}_{(29)}(\varphi, \psi) \leq -\int_{-r}^{0}\left[\left(\frac{2b}{r} - \nu\right)\psi^2(0) - 2L|\psi(0)\psi(\theta)| + \nu\psi^2(\theta)\right]d\theta$$

If $r < b/L$, then one can always find a positive $\nu$ for which the integrand is positive definite in $\psi(0)$, $\psi(\theta)$ which implies $\dot{V}_{(29)}(\varphi, \psi) = 0$ if and only if $\psi = 0$. But, the only invariant set of (29) which lies in the set of $(\varphi, \psi)$ for which $\psi = 0$ is $(0, 0)$. Consequently, the above theorems may be applied to obtain the results that the solution $x = 0$, $y = 0$ of (29) is asymptotically stable and if, in addition $G(x) \to \infty$ as $|x| \to \infty$, this solution is globally asymptotically stable.

<u>Example 6.</u> Consider equation (29) with

$$-f(x)/x > b > 0, \quad g(x)/x > a > 0, \quad |h(x)| < L, \quad r < b/L,$$

and let

$$V(\varphi, \psi) = 2G(\varphi(0)) + \psi^2(0) - \nu \int_{-r}^{0} \left(\int_{\tau}^{0} \psi^2(\theta)d\theta\right)d\tau$$

where $G(x) = \int_0^x g(s)ds$ and $\nu$ is a positive constant to be determined. A direct computation shows that

$$\dot{V}^*_{(29)}(\varphi, \psi) \geq \int_{-r}^{0} [(\frac{2b}{r} - \nu)\psi^2(0) - 2L|\psi(0)| \cdot |\psi(\theta)| + \nu\, \psi^2(\theta)]d\theta$$

0

for some convenient $\nu > 0$ (and in fact, any $\nu < \frac{2b}{r}$) if $r < b/L$ and $\dot{V}^*_{(29)}(\varphi, \psi) = 0$ if and only if $\psi = 0$. But, the only invariant set of (29) for which $\psi = 0$ is $\varphi = 0$. Since $V$ satisfies the conditions of Theorem 4, the origin is unstable.

As a special case of this example, let us consider the Rayliegh equation with retardation,

$$\ddot{x}(t) - \epsilon(1 - \frac{\dot{x}^2(t)}{3})\, \dot{x}(t) + x(t - r) = 0, \quad \epsilon > 0,$$

which, for $r = 0$, can be transformed into the well known van der Pol equation. ~~or, the system~~ If this equation is considered as a system,

$$(30) \qquad \begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= \epsilon\, [1 - \frac{y^2(t)}{3}]y(t) - x(t) + \int_{-r}^{0} y(t + \theta)d\theta, \end{aligned}$$

then, comparison with (29) yields

$$-f(y) = \epsilon\, (1 - \frac{y^2}{3})y$$

$$g(x) = x, \quad h(x) = 1.$$

For any $\gamma$, $0 < \gamma < 1$, if $y^2 < \gamma^2 < 3$, then $-f(y)\,y > \epsilon\,(1 - \gamma^2) > 0$. Consequently, from Theorem 4 and the computation in the above example, it follows that the solution $x = 0$ of (30) is unstable if $r < \epsilon\,(1 - \gamma^2)$. The region $U$ where $V > 0$ consists of all $\varphi$, $\psi$ for which

$$\varphi^2(0) + \psi^2(0) > \nu \int_{-r}^{0} \left(\int_{-\tau}^{0} \psi^2(\theta)d\theta\right)d\tau$$

where $\nu > 0$ can be chosen as small as we like with the only restriction on $\nu$ being that $\nu < \epsilon\,(1 - \gamma^2)/r$. Theorem 4 implies that any solution of (30) with initial function $\varphi \in U \cap C_{\sqrt{3}\gamma}$ must reach the boundary of $C_{\sqrt{3}\gamma}$ in a finite time. If $r = 0$ this is the same result as one obtains for ordinary differential equations; namely, if equation (30) has a cycle it must be outside the set $U \cap C_{\sqrt{3}\gamma}$ if $r < \epsilon\,(1 - \gamma^2)$. This type of result states that the properties of the type mentioned above are continuous in $r$.

Example 7. (Krasovskii 1, p. 1 0 ). Consider the scalar equation

$$(31) \qquad \dot{x}(t) = -\,ax(t) - bx(t - r)$$

where $a > 0$, $b$ are constants. If $x$ is a scalar take $|x|$ as the absolute value of $x$. If

$$V(\varphi) = \frac{1}{2}\varphi^2(0) + \mu\int_{-r}^{0} \varphi^2(\theta)d\theta$$

where $\mu > 0$ is to be determined, then

$$\dot{V}_{(31)}(x_t) = -ax^2(t) - bx(t)x(t-r) + \mu x^2(t) - \mu x^2(t-r)$$

and $\dot{V}_{(31)}(x_t)$ is a negative definite quadratic function of $x(t)$, $x(t-r)$ if

$$(a - \mu)\mu > b^2 .$$

Consequently, if $\mu = a/2$ and $b^2 < a^2$, then the conditions of Corollary 2 are satisfied and the solution $x = 0$ of (31) is globally asymptotically stable (the global nature follows from the linearity).

In the above discussion of (31), the particular Lyapunov function used yielded a stability region which is independent of $r$ and the sign of $b$. The exact region of stability for (31) is easily computed (see Bellman and Cooke [8]) and is indicated in Fig. 3.

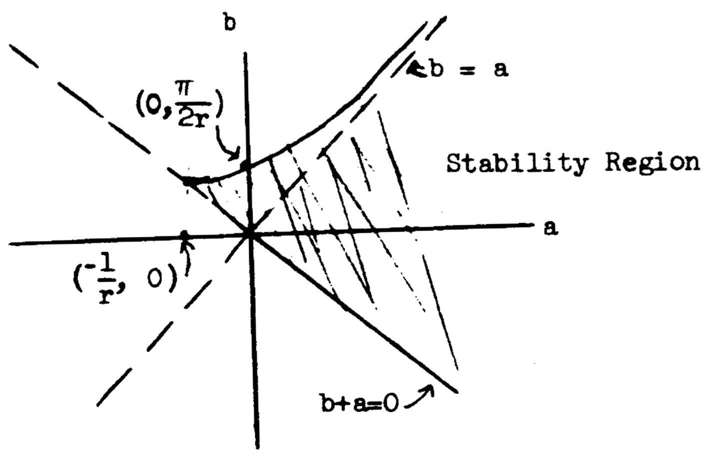


Fig. 3.

The region $|b| < a$ is the maximum region for which stability is assured for all values of $r$, $0 \leq r \leq \infty$. On the other hand, as $r \to 0$ the true region of stability for (6.1) approaches the half-plane $b + a > 0$.

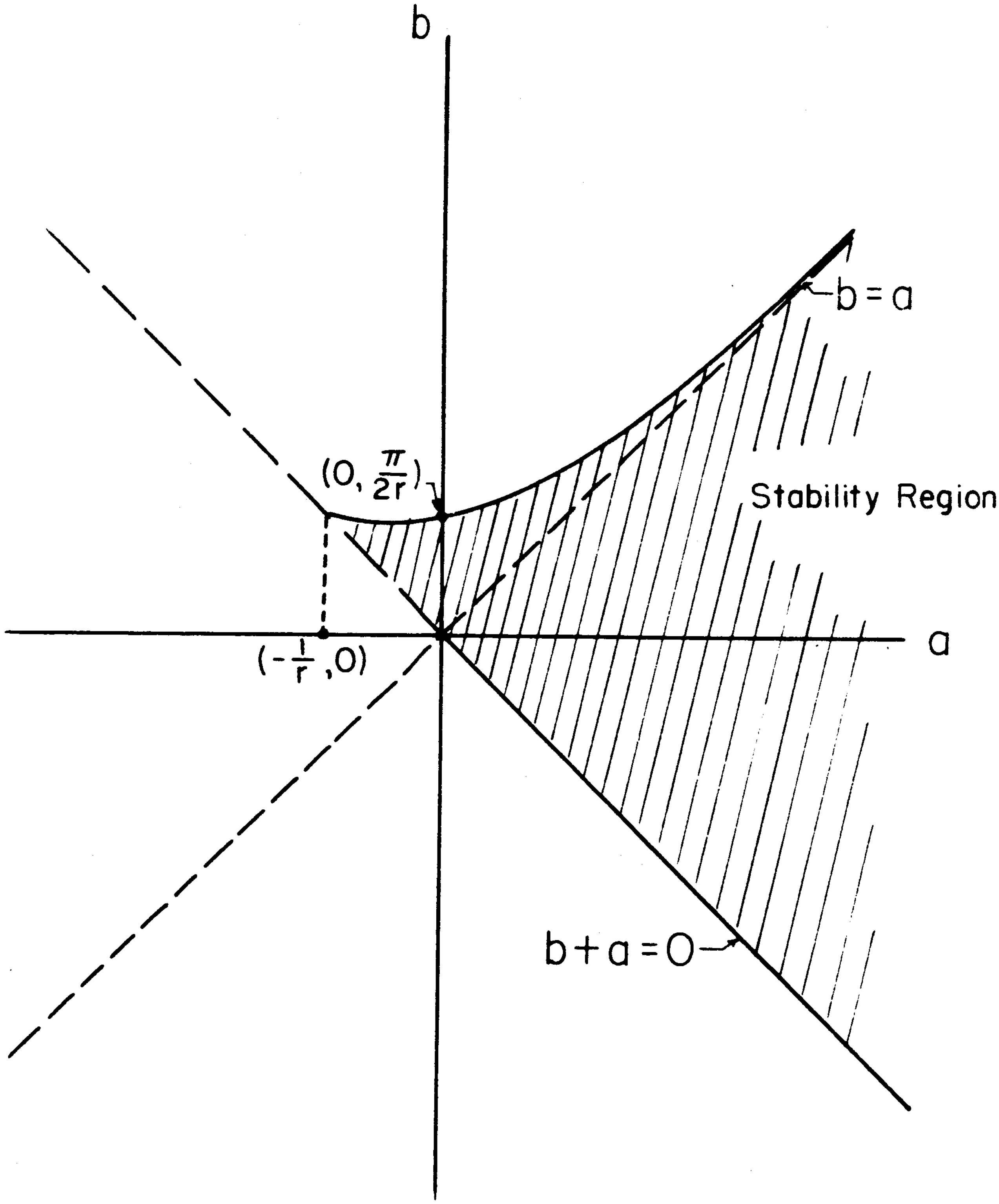


Fig. 3

We ask the following question: is it possible to use a Lyapunov function to obtain a region of stability which depends upon $r$ and has the same qualitative structure as the true region of stability? In particular, can we use a Lyapunov function to obtain a region of stability which approaches the half-plane $b + a > 0$ as $r \to 0$ ?

We proceed as follows. Let $\alpha$ be a constant and $\beta(\theta)$ be a continuously differentiable function of $\theta$ for $-r \leq \theta \leq 0$. If

$$V(x_t) = \frac{x^2(t)}{2} + \alpha x(t) \int_{t-r}^{t} x(u)du + \int_{t-r}^{t} \beta(u - t)x^2(u)du \tag{32}$$

then

$$k|x(t)|^2 \leq V(x_t) \leq K\|x_t\|,$$

if

$$\beta(\theta) > \frac{\alpha^2 r}{2}, \quad -r \leq \theta \leq 0. \tag{33}$$

Furthermore,

$$\dot{V}_{(31)}(x_t) = -\int_{t-r}^{t} [\frac{a - \alpha - \beta(0)}{r} x^2(t) + \frac{1}{r}(\alpha + b)x(t)x(t - r)$$

$$+ \frac{\beta(-r)}{r} x^2(t - r) + \alpha a x(t)x(u)$$

$$+ \alpha b x(t - r)x(u) + \dot{\beta}(u - t)x^2(u)]du.$$

Consequently, if $\beta$ satisfies (33) and the integrand is positive definite in $x(t)$, $x(t - r)$ and $x(u)$, then Corollary 2 will imply the uniform asymptotic stability of the solution $x = 0$ of (31). The necessary and sufficient conditions for the integrand to be positive definite in these variables is that $\alpha$, $\beta$, a, b and c satisfy the following set of inequalities:

$$\Delta_1 \overset{\text{def}}{=} a - \alpha - \beta(0) > 0$$

(34)
$$\Delta_2 \overset{\text{def}}{=} \Delta_1\beta(-r) - \frac{1}{4}(\alpha + b)^2 > 0$$

$$\Delta_3 \overset{\text{def}}{=} \Delta_2\dot{\beta}(\theta) - \frac{\alpha^2 r}{4}[b^2\Delta_1 - b(\alpha + b) + \beta(-r)] > 0$$

If we assume that $\Delta_1 > 0$, $\Delta_2 > 0$ then the previous inequalities will be satisfied if

$$\Delta_1 \overset{\text{def}}{=} a - \alpha - \beta(0) > 0$$

(35)
$$\Delta_2 \overset{\text{def}}{=} \Delta_1\beta(-r) - \frac{1}{4}(\alpha + b)^2 > 0$$

$$\Delta_2\dot{\beta}(\theta) > \frac{\alpha^2 r}{4\Delta_1}\left(b\Delta_1 - \frac{\alpha + b}{2}\right)^2 , \quad -r \leq \theta \leq 0$$

Consequently, if for given $a$, $b$ and $r$ we can choose $\alpha$ and $\beta(\theta)$ so that relations (33) and (35) are satisfied, then the point $(a,b,r)$ will correspond to a set of the parameters for which (31) is asymptotically stable.

Let us check first of all to see whether we can obtain the same results as before for the region of stability which is independent of $r$. For $\alpha = 0$, inequalities (33) and (35) do not depend on $r$ explicitly and reduce to $\beta(\theta) > 0$, $\dot{\beta}(\theta) > 0$, $-r \leq \theta \leq 0$, $a > \beta(0)$, $b^2 < 4(a - \beta(0))\beta(-r)$. Suppose there is a constant $q < 1$ such that $b^2 < qa^2$, $a > 0$. Then we can satisfy the inequalities if we choose $\beta(\theta)$ close enough to $a/2$, which yields the same result as before.

We now make some rather crude estimates of $\alpha$, $\beta$ to obtain some information about that part of the stability region which depends on $r$ and is contained in the region $a + b > 0$. In particular, we want to see if it is possible to obtain the region of stability as the region $a + b > 0$ if $r \to 0$. If we let $\alpha = -b - 2\beta(-r)$ [this maximizes $\Delta_2$], then our inequalities (33), (35) will be satisfied if

$$
(36)\qquad
\begin{aligned}
&a + b > \beta(0)\\
&\dot{\beta}(\theta) > r\,\frac{(b + 2\beta(-r))^2}{4\Delta_1\Delta_2}\,(b\Delta_1 + \beta(-r))^2\\
&\beta(\theta) > \frac{r}{2}(2\beta(-r) + b)^2,\ -r \le \theta \le 0,\\
&\Delta_1 = a + b - (\beta(0) - \beta(-r)),\ \Delta_2 = [a + b - \beta(0)]\beta(-r).
\end{aligned}
$$

We see from these inequalities that as $r \to 0$, the region defined by (36) with $\beta(\theta)$ sufficiently small approaches the region $a + b > 0$, which coincides with the true stability region. It would be interesting to investigate how well the stability retion of (31) can be approximated by a clever choice of the constant $\alpha$ and the function $\beta$ in (32).

Example 8. Consider again the equation

$$\dot{x}(t) = -ax(t) - bx(t - r),$$

where $a + b < 0$ and $r$ is any positive constant. We wish to prove by use of Lyapunov functions that the solution $x = 0$ of this equation is unstable. The exact region of stability for this equation is shown in Fig. 3. The region $a + b < 0$ is the interior of the intersections of the instability regions as a function of $r$.

If $F$ is any given function and

$$V(x_t) = \frac{x^2(t)}{2} - \frac{1}{2}\int_{t-r}^{t} F(t - u)[x(u) - x(t)]^2 du,$$

then it is easily seen that

$$\dot{V}^*(x_t) = \dot{V}(x_t) = -(a+b)x^2(t) - b[x(t-r) - x(t)]\,x(t)$$
$$+ \frac{1}{2} F(r)[x(t-r) - x(t)]^2 - \frac{1}{2}\int_{t-r}^{t} \dot{F}(t-u)[x(u) - x(t)]^2 du$$
$$+ \int_{t-r}^{t} F(t-u)[x(u) - x(t)][-(a+b)x(t) - b[x(t-r)-x(t)]]du.$$

If the expression for $\dot{V}$ is written as an integral from $[t - r, t]$, then the integrand will be a positive definite quadratic form in $x(t)$, $[x(t-r)-x(t)]$, $[x(u) - x(t)]$ if the following inequalities are satisfied:

$$a + b < 0$$
$$\Delta_2 \overset{\text{def}}{=} -(a+b)F(r) - \frac{b^2}{2} > 0$$
$$\Delta_3 \overset{\text{def}}{=} -\Delta_2\dot{F}(\theta) - \frac{(a+b)^2 r}{2} F^2(\theta)F(r) > 0, \quad 0 \leq \theta \leq r.$$

If $a + b < 0$, it is clear that these inequalities can be satisfied by a continuously differentiable positive function $F(\theta)$, $0 \leq \theta \leq r$. Consequently, there exists a positive number $q$ such that

$$\dot{V}^*(\varphi) \geq qr\varphi^2(0), \qquad V(\varphi) \leq \varphi^2(0)/2,$$

for all $\varphi$ in $C$. If $U = \{\varphi \text{ in } C\colon\ \varphi^2(0) > \int_{-r}^{0} F(\theta)[\varphi(\theta) - \varphi(0)]^2 d\theta\}$ then $U$ satisfies i) and ii) of Theorem 4 and the remark after Theorem 4 implies the solution $x = 0$ is unstable.

Example 9. Consider the equation

$$\dot{x}(t) = ax^3(t) + bx^3(t - r)$$

with $a > 0$, $|b| < qa$, $0 < q < 1$ arbitrary. For

$$V(\varphi) = \frac{\varphi^4(0)}{4a} - \frac{1}{2}\int_{-r}^{0} \varphi^6(\theta)d\theta \leq \frac{\varphi^4(0)}{4a}$$

we have

$$\dot{V}^*(\varphi) = \dot{V}(\varphi) = \frac{1}{2}\varphi^6(0) + \frac{b}{a}\varphi^3(0)\varphi^3(-r) + \frac{1}{2}\varphi^6(-r)$$

$$\geq \frac{1}{2}(1 - q)(\varphi^6(0) + \varphi^6(-r))$$

$$\geq \frac{1}{2}(1 - q)\varphi^6(0)$$

and if $U = \{\varphi \in C: \varphi^4(0) > 2a\int_{-r}^{0} \varphi^6(\theta)d\theta\}$, then the same argument as before shows that $x = 0$ is an unstable solution of this equation.

If $a < 0$, $|b| < |a|$, one can choose

$$V(\varphi) = \frac{\varphi^4(0)}{4|a|} + \frac{1}{2}\int_{-r}^{0} \varphi^6(\theta)d\theta$$

and use Corollary 2 to prove the zero solution is stable.

Notice that the same $V$ functionals may be used to show that the zero solution of $\dot{x}(t) = ax^3(t) + bx^4(t = r)$ is stable or unstable according as $a < 0$ or $> 0$, regardless of the size of $b$. One simply must operate in a sufficiently small neighborhood of the origin.

4. <u>Extensions of the theory to the case of infinite lag.</u> In this section, we extend the theory developed in section 2 to the case of infinite lag; that is $r = +\infty$. ~~The application of~~ This theory ~~will~~ can be ~~valid~~ applied to all of the examples of section 3 for which results were obtained independently of $r$. This extension will be clear once an $\omega$-limit set is defined and it is known that an $\omega$-limit set is an invariant set. The compact open topology on the space of continuous functions is employed in this extension and resulted from a conversation with J.P. LaSalle.

Suppose $r$ is a given real number which we allow to be $+\infty$. For any real number $\sigma$, the symbol $[\sigma - r, \sigma]$ denotes the closed interval $\sigma - r \leq t \leq \sigma$ if $r$ is finite and the half open interval $-\infty < t \leq \sigma$ when $r = +\infty$. Let $C = C([-r, 0], E^n)$ be the space of all continuous functions mapping the interval $[-r, 0]$ into $E^n$. The topology on $C$ is taken as the compact open topology which, in this case, is equivalent to uniform convergence on all compact subsets of $[-r, 0]$. If $r$ is finite, then $C$ is a Banach space with the norm of an element $\varphi$ of $C$ given by

$$\|\varphi\| \overset{\text{def}}{=} \|\varphi\|_{[-r,\, 0]} = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)| .$$

If $r = +\infty$, then $C$ is not a Banach space, but a complete metric space with a metric $\rho$ that can be defined as

$$\rho(\varphi, \psi) = \sum_{N=0}^{\infty} m_N ,$$

$$m_N = \min(2^{-N}, \|\varphi - \psi\|_{[-(N+1),\, -N]}) .$$

Notice that this metric has the important property that

(37) $\|\varphi\|_{[-(N+1),\ -N]} \leq \rho(\varphi, 0), \quad N = 0, 1, 2, \ldots$

One can show that a sequence $\varphi_n \to \varphi$ [that is, $\rho(\varphi_n, \varphi) \to 0$ as $n \to \infty$] if and only if for every nonnegative integer $N$, $\|\varphi_n - \psi\|_{[-N,\ 0]} \to 0$ as $n \to \infty$. For more details on this space of functions, see Arens [9], Bourbaki [10], Kelley [11].

If $H$ is a positive constant, we use the notation $C_H$ to denote the set $\{\varphi$ in $C$: $\rho(\varphi, 0) < H\}$; that is, the open "ball" in $C$ with center at $0$ and radius $H$.

If $\sigma$ is any real number and $x$ is any continuous function with domain $[\sigma - r, \infty)$ and range in $E^n$, we let $x_t$, $t \geq \sigma$, denote the restriction of $x$ to the interval $[t - r, t]$; that is, $x_t$ belongs to $C$.

If $f(\varphi)$ is a function defined for every $\varphi$ in $C_H$ and $\dot{x}(t)$ is the right hand derivative of $x(t)$, we consider the following autonomous functional differential equation:

(38) $\dot{x}(t) = f(x_t).$

A solution is defined the same way as for the case of finite lag. Relation (37) allows one to prove the following existence and uniqueness theorem: If $f(\varphi)$ is continuous in $C_H$, then for any $\varphi$ in $C_H$, there is a solution of (1) with initial condition $\varphi$ at $t = 0$. If $f(\varphi)$ is locally Lipschitzian in $\varphi$; that is, for any $H_1 < H$, there exists an $L_{H_1}$ such that

$$|f(\varphi) - f(\psi)| \leq L_H \rho(\varphi, \psi),$$

then there is only one solution with initial condition $\varphi$ at $t = 0$ and the solution $x(\varphi)$ depends continuously upon $\varphi$. Also, $f(\varphi)$ locally Lipschitzian in $\varphi$ implies the solutions can be extended in $C$ until the boundary of $C_H$ is reached.

Stability and asymptotic stability are defined the same way as for the finite lag, except use is made of the metric $\rho$. Relation (37) implies that the definitions made in this manner yield the desired properties.

In the remainder of the discussion we assume $f(\varphi)$ is continuous and locally Lipschitzian in $C_H$ and the solution $x(\varphi)$ of (38) with initial condition $\varphi$ at $t = 0$ is defined on $[-r, \infty)$. An element $\psi$ of $C$ is in $\Omega(\varphi)$, the $\omega$-limit set of $\varphi$, if there is a sequence of nonnegative real numbers $t_n$, $t_n \to \infty$ as $n \to \infty$, such that $\rho(x_{t_n}(\varphi), \psi) \to 0$ as $n \to \infty$. A set $M$ is said to be invariant if for any $\psi$ in $M$ and for any $\sigma$ in $(-\infty, 0]$, the solution $x_t(\sigma, \psi_\sigma)$ of (38) with initial value $\psi_\sigma$ at $t = \sigma$ is defined for $t \geq \sigma$, $x_t(\sigma, \psi_\sigma)$ belongs to $M$ for $t \geq \sigma$ and $x_0(\sigma, \psi_\sigma) = \psi$. Notice that all solutions of (38) on an invariant set must be defined on $(-\infty, \infty)$.

Lemma 3. If $\varphi$ in $C_H$ is such that the solution $x = x(\varphi)$ of system (38) with initial function $\varphi$ at $t = 0$ is defined on $[-r, \infty)$ and $\rho(x_t, 0) \leq H_1 < H$ for $t$ in $[0, \infty)$, then $\Omega(\varphi)$ is a nonempty, compact, connected, invariant set and $\rho(x_t, \Omega(\varphi)) \to 0$ as $t \to \infty$.

Proof: If $\rho(x_t, 0) \leq K$, $t \geq 0$, then there exists a constant $M$ such that $|x(t)| \leq K$, $|\dot{x}(t)| \leq M$ for all $t \geq 0$. For any nonnegative integer $N$, let $\delta_N = [-N, 0] \cap [-r, 0]$. The restriction of $x$ to the interval $[t-N, t]$, $t \geq 0$, belong to a compact subset of $C(\delta_N, E^n)$. In $C(\delta_N, E^n)$, we let $\|\varphi\|_{\delta_N}$ denote the norm which is given by $\|\varphi\|_{\delta_N} = \max_{\theta \text{ in } \delta_N} |\varphi(\theta)|$. Consequently, there exists a sequence $t_n = t_n(N)$, $t_n \to \infty$ as $n \to \infty$ and a function $\psi$ in $C(\delta_N, E^n)$ such that $\|x_{t_n} - \psi\|_{\delta_N} \to 0$ as $n \to \infty$. Application of the diagonalization procedure yields a sequence $s_n$, independent of $N$, and a function $\psi$ in $C([-r, 0], E^n)$ such that

$$\|x_{s_n} - \psi\|_{\delta_N} \to 0 \quad \text{as} \quad n \to \infty$$

for all integers $N = 0, 1, 2, \ldots$ . This proves $\Omega(\varphi)$ is nonempty.

$\Omega(\varphi)$ is obviously bounded. To show $\Omega(\varphi)$ is closed, suppose $\psi_n$ in $\Omega(\varphi)$ and $\psi_n \to \psi$ as $n \to \infty$. There exists a sequence of real numbers $t_n$, which can be chosen independent of $\psi_n$ by the diagonalization procedure, such that for any $\epsilon > 0$, there is an integer $N_o = N_o(\epsilon)$ such that

$$\rho(x_{t_n}(\varphi), \psi_n) < \epsilon/2, \quad \rho(\psi, \psi_n) < \epsilon/2, \quad n \geq n_o.$$

Therefore, $\rho(x_{t_n}(\varphi), \psi) < \epsilon$ for $n \geq N_o$ which shows that $\psi$ is in $\Omega(\varphi)$.

To show $\Omega(\varphi)$ is compact, suppose $\psi_n$ in $\Omega(\varphi)$, $n = 1, 2, \ldots$ . It follows from the first part of the proof of the lemma, that $\psi_n$ restricted to $[-N, 0) \cap [-r, 0]$ belongs to a compact subset of $C(\delta_N, E^n)$. Consequently, one applies the diagonalization procedure to show the existence of a $\psi$ in $C([-r, 0], E^n)$ and a subsequence $n_k \to \infty$ as $k \to \infty$ such that $\rho(\psi_{n_k}, \psi) \to 0$ as $k \to \infty$.

To show $\Omega(\varphi)$ is invariant, suppose $\psi$ is in $\Omega(\varphi)$ and the sequence $t_n \to \infty$ as $n \to \infty$ is such that $\rho(x_{t_n}(\varphi), \psi) \to 0$ as $n \to \infty$. For any integer $N$ and any $\tau$ in $[-N, N]$, choose $n_o(N)$ so large that $t_n + \tau \geq 0$, $-N \leq \tau \leq N$. By an argument similar to that used in the first part of the proof of this lemma, there exists a subsequence $s_n$ of the $t_n$ and a $g_{\tau,N}(\varphi)$ which belongs to $C(\delta_N, R^n)$ for each $\tau$ in $[-N, N]$ and

$$\|x_{s_n + \tau}(\varphi) - g_{\tau,N}(\varphi)\|_{\delta_N} \to 0$$

as $n \to \infty$ uniformly for $\tau$ in $[-N, N]$. By use of the diagonalization procedure, one shows there exists another subsequence $p_n$, independent of $N$, a $g_\tau(\varphi)$ which belongs to $C([-r, 0], E^n)$ for each $\tau$ in $(-\infty, \infty)$ such that

$$\|x_{p_n + \tau}(\varphi) - g_\tau(\varphi)\|_{\delta_N} \to 0 \quad \text{as} \quad n \to \infty$$

for every integer $N$ and all $\tau$ in $(-\infty, \infty)$.

It is easy to show that $g_\tau(\varphi)$ satisfies (38) for any $\tau$ in $(-\infty, \infty)$ and it is obvious that $g_0(\varphi) = \psi$. Therefore, $\Omega(\varphi)$ is invariant. $\Omega(\varphi)$ is obviously connected and the lemma is proved.

The remainder of the results of section 2 are verified exactly as before to obtain sufficient conditions for stability and instability for the case of infinite time lag.

REFERENCES:

1. Krasovskii, N. N., Stability of motion. Moscow, 1959; Stanford Univ. Press, 1963.

2. LaSalle, J. P., The extent of asymptotic stability. Proc. Nat. Acad. Sc., 46(1960), 363-365.

3. Hale, J. K., A stability theorem for functional-differential equations. Proc. Nat. Acad. Sci., 50(1963), 942-946.

4. Levin, J. J. and J. Nohel, On a nonlinear delay equation. J. Math. Ana. Appl. 8(1964), 31-44.

5. Ergen, W. K., Kinetics of the circulating fuel nuclear reactor. J. Appl. Physics, 25(1954), 702-711.

6. Volterra, V., Sur la théorie mathématique des phénomènes héréditaires. J. Math. Pures Appl., 7(1928), 249-298.

7. Volterra, V., Théorie Mathématique de la Lutte pour la Vic. Gauthier-Villars, 1931.

8. Bellman, R. and K. Cooke, Differential-difference equations. Academic Press, 1963.

9. Arens, R. F., A topology for spaces of transformations, Ann. Math. 47(1946), 480-495.

10. Bourbaki, N., Espaces Vectoriels Topologiques, livre 5. Herman, Paris, 1953.

11. Kelley, J. L., General Topology. Van Nostrand, New York, 1955.